LAMINAR WEAKLY SWIRLED JET PROPAGATING ALONG A STRAIGHT CIRCULAR CONE

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The propagation of a laminar, nonisothermal, weakly swirled liquid jet along the nonconducting surface of a straight circular cone is examined. An explanation is given of the influence of the temperature dependence of the viscosity (assumed to be linear) on the hydrodynamic characteristics of the jet.

The laws of propagation of a jet source over the surface of a cone with various boundary conditions for the temperature and constant physical properties of the liquid were examined in [1].

In the present paper we present the results of solving a similar problem for a swirled jet of liquid with viscosity varying in the flow field. All the other properties of the liquid (density, thermal conductivity, etc.) are considered constant.

For a liquid the dependence of viscosity on temperature is very complex. For some liquids this dependence may be represented in the form proposed by Reynolds [2, 3]:

$$\mu = \mu_\infty \exp(-b\,\Delta T),$$

while for others the hyperbolic relation

$$\mu = \mu_\infty (1 + b\,\Delta T)^{-1}$$

may be used [4, 5]. In the case of mildly nonisothermal flow (this is the case examined below) each of the expressions for the viscosity is simplified and may be represented by a linear relation:

$$\mu = \mu_\infty (1 - b\,\Delta T), \quad \Delta T = T - T_\infty. \tag{1}$$

The flow of a weakly swirled nonisothermal liquid jet over the thermally insulating surface of a cone is described by the following system of equations of the laminar boundary layer: *

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\nu \frac{\partial u}{\partial y}\right), \tag{2}$$

$$\rho \frac{w^2}{x}\operatorname{ctg}\vartheta = \frac{\partial p}{\partial y}, \tag{3}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{uw}{x} = \frac{\partial}{\partial y}\left(\nu \frac{\partial w}{\partial y}\right), \tag{4}$$

$$\frac{\partial}{\partial x}(xu) + \frac{\partial}{\partial y}(xv) = 0, \tag{5}$$

$$u\frac{\partial \Delta T}{\partial x} + v\frac{\partial \Delta T}{\partial y} = a\frac{\partial^2 \Delta T}{\partial y^2} \tag{6}$$

with boundary and integral conditions

$$u = v = w = 0, \quad \frac{\partial \Delta T}{\partial y} = 0 \quad \text{when } y = 0,$$

$$u = w = 0, \quad \Delta T = 0 \quad \text{when } y = \infty; \tag{7}$$

$$\int_0^\infty \rho x u^2 \left(\int_0^y \rho x u dy\right) dy + \int_0^x \left(\int_0^\infty \rho^2 x^2 \nu u \frac{\partial u}{\partial y} dy\right) dx = K = \text{const}; \tag{8}$$

$$\int_0^\infty \rho x^2 u w \left(\int_0^y \rho x u dy\right) dy + \int_0^x \left(\int_0^\infty \rho^2 \nu x^3 u \frac{\partial w}{\partial y} dy\right) dx = N = \text{const}; \tag{9}$$

$$2\pi \sin\vartheta \int_0^\infty \rho C_p u \Delta T x dy = Q = \text{const}. \tag{10}$$

We will introduce the small parameter $\omega = bQ/\lambda L$ ($L = K/\rho^2\nu^3_\infty$), and represent the components of velocity and pressure in the form of series in terms of this small parameter:

$$\begin{aligned} u &= u_0 + \omega u_1 + \omega^2 u_2 + \cdots, \\ v &= v_0 + \omega v_1 + \omega^2 v_2 + \ldots, \\ w &= w_0 + \omega w_1 + \omega^2 w_2 + \cdots. \end{aligned} \tag{11}$$

We also rewrite (1) as

$$\nu = \nu_\infty \left(1 - \omega \frac{L\lambda}{Q} \Delta T\right). \tag{12}$$

Substituting the expansions (11), in addition to (12), into the system (2)–(5) and equating coefficients of the same powers of ω on the left and right sides of the equations obtained, we have the following systems of zeroth, first, and second order approximations:

$$u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} = \nu_\infty\frac{\partial^2 u_0}{\partial y^2},$$

$$u_0\frac{\partial w_0}{\partial x} + v_0\frac{\partial w_0}{\partial y} + \frac{u_0 w_0}{x} = \nu_\infty\frac{\partial^2 w_0}{\partial y^2}, \tag{13}$$

$$\frac{\partial}{\partial x}(xu_0) + \frac{\partial}{\partial y}(xv_0) = 0,$$

*Strictly speaking, equation (2) for a swirled jet should be written in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{w^2}{x} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right).$$

However, for a weakly swirled jet at a large distance from the nozzle, the terms w^2/x and $\partial p/\partial x$ will be much smaller than the other terms of the equation, as follows from the solution obtained below. Therefore, as was done also in the other cases of weakly swirled jets [6, 7], these small terms are omitted in Eq. (2).

$$u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_0}{\partial y} + v_0 \frac{\partial u_1}{\partial y} = = \nu_\infty \frac{\partial}{\partial y}\left[\frac{\partial u_1}{\partial y} - \frac{\lambda L}{Q}\Delta T \frac{\partial u_0}{\partial y}\right], \quad (14)$$

$$u_1 \frac{\partial w_0}{\partial x} + u_0 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_0}{\partial y} + v_0 \frac{\partial w_1}{\partial y} + \frac{u_0 w_1 + u_1 w_0}{x} = = \nu_\infty \frac{\partial}{\partial y}\left[\frac{\partial w_1}{\partial y} - \frac{\lambda L}{Q}\Delta T \frac{\partial w_0}{\partial y}\right], \quad (15)$$

$$\frac{\partial}{\partial x}(x u_1) + \frac{\partial}{\partial y}(x v_1) = 0. \quad (16)$$

The system of equations of the zeroth order approximation (13), with the boundary and integral conditions

$$u_0 = v_0 = w_0 = 0 \quad \text{when } y = 0,$$

$$u_0 = w_0 = 0 \quad \text{when } y = \infty$$

$$\int_0^\infty \rho x u_0^2 \left(\int_0^y \rho x u_0 dy\right) dy = K;$$

$$\int_0^\infty \rho x^2 u_0 w_0 \left(\int_0^y \rho u_0 x dy\right) dy = N$$

has the solution [1]

$$u_0 = A x^\alpha F'(\varphi), \quad v_0 = -\frac{A}{B} x^{\alpha-\beta-1} \times \times [(\alpha - \beta + 1) F + \beta\varphi F'],$$

$$w_0 = C x^\varepsilon \Phi(\varphi), \quad p_\infty - p = D x^\delta P(\varphi), \quad \varphi = B y x^\beta,$$

$$\alpha = -\frac{3}{2}, \quad \beta = -\frac{5}{4}, \quad \varepsilon = -\frac{5}{2}, \quad \delta = -\frac{19}{4},$$

$$F' = \Phi = \frac{2}{3}(F_\infty^{3/2} F^{1/2} - F^2), \quad F_\infty = 1.7818,$$

$$P(\varphi) = -\int_\infty^\varphi \Phi^2 d\varphi, \quad A = \sqrt{\frac{3K}{4\rho^2 \nu_\infty I_1}},$$

$$B = \sqrt[4]{\frac{27K}{64\rho^2 \nu_\infty^3 I_1}},$$

$$C = \frac{N}{\rho}\sqrt{\frac{3}{4\nu_\infty K I_1}}, \quad D = \left(\frac{N}{K}\right)^2 \operatorname{ctg}\vartheta \sqrt[4]{\frac{3K^3}{4\rho^2 \nu_\infty I_1^3}},$$

$$\left(I_1 = \int_0^\infty F (F')^2 d\varphi\right). \quad (17)$$

The solution of the energy equation (6) with account for expressions (17) and the boundary conditions for temperature (7) is well-known [6]:

$$\Delta T = \Gamma x^\gamma \Theta(\varphi); \quad \gamma = -\frac{3}{4},$$

$$\Gamma = \frac{Q}{2\pi\rho C_p \sin\vartheta} \sqrt[4]{\frac{3 I_1}{4\rho^2 \nu_\infty K}} \left[\int_0^\infty F' \Theta d\varphi\right]^{-1},$$

$$\Theta = \exp\left(-\sigma \int_0^\varphi F d\varphi\right); \quad \sigma = \frac{\nu_\infty}{a}. \quad (18)$$

To integrate the first-approximation system of equations (14)–(16), we rewrite the components of velocity and pressure in the form

$$u_1 = A_1 x^{\alpha_1} f(\varphi),$$

$$w_1 = C_1 x^{\varepsilon_1} \tilde{F}(\varphi),$$

$$p_1 = D_1 x^{\delta_1} \Pi(\varphi). \quad (19)$$

Substitution of these expressions into (14)–(16) gives the following system of ordinary differential equations:

$$f'' + F f' + 5F' f = (\Theta F'')', \quad (20)$$

$$\tilde{F}'' + \tilde{F}' F + 3F' \tilde{F} = (\Theta\Phi')' - 2f'\Phi, \quad (21)$$

$$\Pi' = -2\Phi\tilde{F}. \quad (22)$$

In writing down these equations it was assumed that

$$A_1 = \frac{\lambda L}{Q}\Gamma A, \quad C_1 = \frac{\lambda L}{Q}\Gamma C;$$

$$D_1 = \frac{\lambda L}{Q}\Gamma D, \quad \alpha_1 = \frac{-9}{4}, \quad \varepsilon_1 = -\frac{13}{4}; \quad \delta_1 = -\frac{11}{2}.$$

The functions f, $\tilde{F}$ and Π must satisfy the following boundary and integral conditions:

$$f(0) = \tilde{F}(0) = f(\infty) = \tilde{F}(\infty) = \Pi(\infty) = 0,$$

$$\int_0^\infty (2FF'f - F''f + F'F''\Theta - F'f')\, d\varphi + \int_0^\infty (F')^2 \left(\int_0^\varphi f d\varphi\right) d\varphi = 0,$$

$$\int_0^\infty (FF'\tilde{F} + Ff\Phi - f\Phi' + F'\Phi'\Theta - F'\tilde{F}')\, d\varphi + \int_0^\infty F'\Phi\left(\int_0^\varphi f d\varphi\right) d\varphi = 0. \quad (23)$$

It is known [1] that $\Phi = F'$, and therefore, assuming that $f = F$, we may transform Eq. (21) to the form (20). Bearing in mind the identity of the boundary conditions for the functions f and F, we come to the conclusion that, to solve the problem as formulated, it is sufficient to integrate Eq. (20). Introducing the new independent variable $t = \sqrt{F/F_\infty}$, we rewrite Eq. (20) and the boundary conditions (23) in the form

$$(1 - t^3) f' + 30 t f = \frac{2}{3} F_\infty^2 \frac{d}{dt}\left[(1 - t^3)^{\sigma+1}(1 - 4t^3)\right], \quad (24)$$

$$f(0) = f(1) = 0. \quad (25)$$

The general solution of (24) has the form

$$f = (1 - 4t^3)(1 - t^3)\left[C_1 + C_2\left(\frac{t}{1 - t^3} + \frac{16t}{1 - 4t^3} - \frac{5}{3}\ln\frac{(1-t)^2}{1 + t + t^2} + \frac{10}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}\right)\right] + f_0, \quad (26)$$

where f_0 is a particular solution of the inhomogeneous equation (24), which may easily be obtained by ordinary methods, for a specific Prandtl number σ. Thus, for example, for $\sigma = 3$

$$f_0 = \frac{2}{3} F_\infty^2 \left(\frac{10}{21} t^{13} - \frac{227}{105} t^{10} + \right. \quad (27)$$

$$+\frac{67}{14}t^7-\frac{23}{6}t^4+\frac{11}{15}t\Big), \qquad (27)\ \text{cont'd}$$

and for $\sigma = 5$

$$f_0=\frac{2}{3}F_\infty^2\Big(\frac{7}{26}t^{19}-\frac{1551}{910}t^{16}+\frac{2937}{637}t^{13}-\frac{1718}{245}t^{10}+$$

$$+\frac{363}{49}t^7-\frac{30}{7}t^4+\frac{5}{7}t\Big). \qquad (28)$$

For the above values of σ, $f_0(0) = f_0(1) = 0$, and it therefore follows from the boundary conditions that $C_1 = C_2 = 0$.

We may verify by a direct check that the functions found are the desired solution of the problem and also satisfy the integral condition.

Taking account of Eqs. (17), (19), and (20), we obtain an expression for the dimensionless components of the longitudinal velocity and of the swirl velocity in the form

$$\frac{u}{u_{m0}}=\frac{w}{w_{m0}}=\frac{2}{3}F_\infty^2\,[t-t^4+\Omega\times$$

$$\times\Big(\frac{11}{15}t-\frac{23}{6}t^4+\frac{67}{14}t^7-\frac{227}{105}t^{10}+\frac{10}{21}t^{13}\Big)\Big];$$

$$\Omega=\omega\frac{\lambda L\Gamma}{Q}x^{-3/4}. \qquad (29)$$

Here, $u_{m0} = Ax^{-3/2}$ and $w_{m0} = Cx^{-5/2}$ are the maximum values of the respective velocity components in an isothermal jet.

It may be seen from the figure that in a mildly non-isothermal jet of incompressible liquid, calculation of the temperature dependence of the viscosity is considerably affected by the pressure field. As in other flow cases [8], the maximum velocity in a hot jet approaches the surface washed. The effective jet thickness is diminished, and the friction stress at the surface of the cone is increased. The latter follows from analysis of the expression

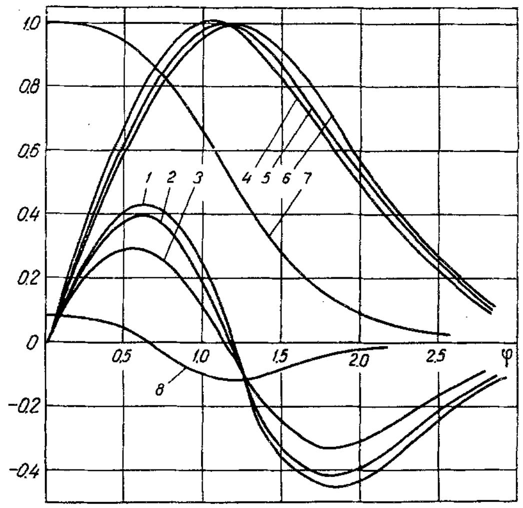
$$\frac{\tau_w}{\tau_{w0}}=\frac{\mu_w}{\mu_\infty}\Big(1+\frac{11}{15}\omega\Big),$$

where τ_{w0} is the stress at the surface in isothermal flow over a cone ($T_w = T_\infty$).

For a cold jet everything is reversed.

NOTATION

u, v, w are the longitudinal, transverse, and circular velocity components; ρ is the density; T is the temperature, μ, ν, a, λ are the dynamic and kinetic viscosities, and thermal diffusivity and conductivity; and conductivity; α, β, δ, ε are the constants in the similarity transformations; A, B, C, D, Γ are constants; $\varphi = Bx^\beta y$ is a reduced coordinate; F, f, Φ, $\tilde{F}$, Π, Θ are the dimensionless functions of the coordinate φ; $\sigma = \nu_\infty/a$ is the Prandtl number; ϑ cone angle; N, K, Q are the constants in the integral conditions; τ_w is the friction stress at the wall; ω, Ω are small parameters. Subscripts: w is value at the wall; ∞ is value at an infinite distance from the wall.



The function $f(\varphi)$ for values of the Prandtl number σ of 3 (1), 5 (2), and 10 (3), profiles of dimensionless velocity components u/u_{m0}, w/w_{m0} for a hot non-isothermal jet with $\Omega = 0.1$ (4), for an isothermal jet with $\Omega = 0$ (5), and for a cold jet with $\Omega = -0.1$ (6), the profile of dimensionless pressure $(p - p_\infty)/(p_w - p_\infty)$ in isothermal flow (7), and the profile of dimensionless pressure $\Pi/(p_w - p_\infty)$ with $\sigma = 3$ (8).

REFERENCES

1. L. A. Vulis and V. P. Kashkarov, ZhTF, no. 12, 1956.

2. M. Raffi, J. R. Turian, and B. Buron, Chem. Eng. Sci., 28, no. 11, 1963.

3. S. A. Bostandzhiyan, A. G. Merzhanov, and S. I. Khudyaev, PMTF [Journal of Applied Mechanics and Technical Physics], no. 5, 1965.

4. A. K. Pavlin, PMM, 29, no. 5, 1955.

5. S. A. Regirer, PMM, 21, no. 3, 1957.

6. L. A. Vulis and V. P. Kashkarov, Theory of Viscous Fluid Jets [in Russian], Izd. "Nauka," Moscow, 1965.

7. L. G. Loitsyanskii, Tr. Leningradskogo politekhnicheskogo in-ta im. M. I. Kalinina (tekhnicheskaya gidromekhanika), no. 5, 1953.

8. V. P. Kashkarov and B. M. Mikhaelyan, Problems of Thermal Power Engineering and Applied Thermophysics [in Russian], no. 3, Izd. "Nauka," Alma-Ata, 1966.

13 May 1966 Kazakh University, Alma-Ata